ISRAEL JOURNAL OF MATHEMATICS **128** (2002), 1–27

ISOMETRIC APPROXIMATION PROPERTY IN EUCLIDEAN SPACES

BY

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ABSTRACT

We give a necessary and sufficient quantitative geometric condition for a compact set $A \subset \mathsf{R}^n$ to have the following property with a given $c \geq 1$: For every $\varepsilon > 0$ and for every map $f\colon A \to \mathsf{R}^n$ such that

$$\Big| |fx - fy| - |x - y| \Big| \leq \varepsilon \quad \text{for all } x, y \in A,$$

there is an isometry $S\colon A \to \mathsf{R}^n$ such that $|Sx - fx| \leq c\varepsilon$ for all $x \in A$.

1. Introduction

1.1. *Nearisometries.* Let X and Y be metric spaces, in which the distance between points a and b is written as $|a-b|$. A map $f\colon X \to Y$ is a **nearisometry** if there is $\varepsilon \geq 0$ such that

$$|x - y| - \varepsilon \leq |fx - fy| \leq |x - y| + \varepsilon$$

for all $x, y \in X$. More precisely, we say that such a map is an ε-**nearisometry**. We do not assume that f is continuous. In the literature, the ε-nearisometries are often called ε-isometries.

Suppose that $A \subset \mathsf{R}^n$. For $c \geq 1$, we say that the set A has the c-**isometric approximation property**, abbreviated c-IAP, if for each $\varepsilon > 0$ and for each ε-nearisometry $f\colon A \to \mathsf{R}^n$ there is an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $\|S - f\|_A \leq c\varepsilon$, where we use the notation $\|g\|_A = \sup\{|gx|\colon x \in A\}$.

Received January 31, 2001

It follows from the classical result of Hyers–Ulam [HU, Th. 4] that the whole space R^n has the 10-IAP. In fact, a surjective ε-nearisometry $f\colon E \to F$ between Banach spaces E and F can be approximated by a surjective isometry $S\colon E \to F$ with $\|S - f\|_E \le 2\varepsilon$; see [OŠ, p. 620] or [BL, 15.2]. For maps between Hilbert spaces this holds with the bound $\sqrt{2}\varepsilon$; see [Še, 1.3]. The surjectivity condition is unnecessary in finite-dimensional spaces; see [BŠ, Th. 1]. Thus the whole space R^n has the $\sqrt{2}$-IAP.

In this paper we consider the case where A is a **bounded** subset of R^n. This case is essentially different from the case $A = \mathsf{R}^n$, in which the proof is based on the behavior of f near the point at infinity. In fact, we shall always assume that A is **compact** in R^n. This is no loss of generality, since every ε-nearisometry $f\colon A \to \mathsf{R}^n$ of a bounded set $A \subset \mathsf{R}^n$ can be extended to an ε-nearisometry $g\colon \bar{A} \to \mathsf{R}^n$ by choosing for each $b \in \bar{A} \setminus A$ a sequence (x_n) in A converging to b such that $f(x_n)$ converges to some point $g(b) \in \mathsf{R}^n$.

In [ATV] we gave a sufficient condition for the c-IAP in terms of **thickness**. Let $e \in \mathsf{R}^n$ with $|e| = 1$, and let $\pi_e\colon \mathsf{R}^n \to \mathsf{R}$ be the projection $\pi_e x = x \cdot e$. The thickness of a bounded set $A \subset \mathsf{R}^n$ is the number

$$\theta(A) = \inf\{d(\pi_e A)\colon |e| = 1\},$$

where d denotes diameter. Then $0 \le \theta(A) \le d(A)$. In [ATV, 3.3] we proved that if $\theta(A) \ge q d(A) > 0$, then A has the c-IAP with $c = c_n/q$ where c_n depends only on n.

It will follow from Theorem 2.7 of the present paper that, conversely, if A has the c-IAP for some c and if A contains more than n points, then $\theta(A) > 0$. However, this result is not quantitative: the c-IAP does not give any upper bound for $d(A)/\theta(A)$. For example, let $t > 0$ and let $A = \{0, e_1, te_2\} \subset \mathsf{R}^2$. A straightforward proof shows that A has the 8-IAP while $\theta(A) < t$ and $d(A) = \sqrt{1+t^2}$.

The purpose of this paper is to give a **quantitative** geometric characterization for compact sets $A \subset \mathsf{R}^n$ with the c-IAP. In Section 2 we define the concept of a c-**solar system**, and in the rest of the paper we prove that A is a c-solar system if and only if , quantitatively, A has the c-IAP.

We remark that all compact sets $A \subset \mathsf{R}^n$ have the following property [ATV, 2.2]: Let $f\colon A \to l_2$ be an $\varepsilon d(A)$-nearisometry with $\varepsilon \le 1$. Then there is an isometry $S\colon \mathsf{R}^n \to l_2$ such that $\|S - f\|_A \le c_n d(A)\sqrt{\varepsilon}$, where c_n depends only on n.

ACKNOWLEDGEMENT: I thank D. A. Trotsenko for suggesting the problem by calling my attention to the properties of the sets

$$\{0, e_1, te_2\} \quad \text{and} \quad \{0, e_1, te_2, e_1 + te_2\}$$

in the plane. I also thank him and P. Alestalo for careful reading of the manuscript, for valuable remarks and for pleasant collaboration in the related work [ATV].

1.2. *Notation.* The standard basis of the euclidean n-space R^n is written as $(e_1, \ldots, e_n)$. If $0 \le k \le n$, we identify the space R^k with the linear subspace of R^n generated by $e_1, \ldots, e_k$. We set $\mathsf{R}^n_+ = \{x \in \mathsf{R}^n \colon x_n \ge 0\}$. The distance between nonempty sets $A, B \subset \mathsf{R}^n$ is written as $d(A, B)$. Furthermore, $d(A)$ is the diameter of A, and aff A is the affine subspace generated by A. For $x \in \mathsf{R}^n$ and $1 \le k \le n$ we set

$$x_{k*} = d(x, \mathsf{R}^{k-1}) = \sqrt{x_k^2 + \cdots + x_n^2}.$$

Then

$$x = \sum_{i=1}^{k-1} x_i e_i + x_{k*} e,$$

where $e = e(x, k)$ is a unit vector perpendicular to R^{k-1}.

We let $\bar{B}(x, r)$ denote the closed ball in R^n with center x and radius r, and we abbreviate $\bar{B}(r) = \bar{B}(0, r)$ and $\bar{B}^n = \bar{B}(1)$. To simplify notation, we often omit parentheses writing $fx = f(x)$ etc. For real numbers s, t we write $s \vee t = \max\{s, t\}$, $s \wedge t = \min\{s, t\}$.

1.3. *Convention.* To avoid trivialities, we shall always assume without further notice that the set $A \subset \mathsf{R}^n$ contains at least two points.

2. Solar systems

2.1. *Maximal sequences.* Let $A \subset \mathsf{R}^n$ be compact. A finite sequence $\bar{u} = (u_0, \ldots, u_m)$ of points in A is said to be a **maximal sequence** in A if, setting $E_k = \text{aff}\{u_0, \ldots, u_k\}$, the number

$$h_k = h_k(\bar{u}) = d(u_k, E_{k-1}) \tag{2.2}$$

is maximal in A for all $1 \le k \le m$, that is, $d(x, E_{k-1}) \le h_k$ for all $x \in A$. If $\dim \text{aff}\, A = k < m$, we also assume that $u_j = u_0$ for $k + 1 \le j \le m$. Observe that $A \subset \bar{B}(u_0, |u_1 - u_0|)$.

If $\bar{u}$ is a maximal sequence in A, then

$$|u_1 - u_0| = h_1 \geq \cdots \geq h_m \geq 0,$$

and $h_m > 0$ if and only if $\dim \operatorname{aff} A \geq m$.

Given a point $a \in A$, there always exists a maximal sequence $\bar{u}$ in A with $u_0 = a$, but this sequence is not always unique. A maximal sequence $\bar{u}$ is said to be **normalized** if $u_0 = 0, u_1 = e_1$, and $u_k \in \mathsf{R}^k_+$ for all $2 \leq k \leq m$. Then $E_k = \mathsf{R}^k$ for all $k \leq \dim \operatorname{aff} A$, and $h_k = (u_k)_k$. Given a maximal sequence $\bar{u}$ in A, there is a similarity $T\colon \mathsf{R}^n \to \mathsf{R}^n$ such that the sequence $T\bar{u}$ is normalized and maximal in TA.

Observe that u_0 is an arbitrary point of A. In this respect, the definition above differs from the definition of a maximal sequence in [ATV], where we assumed that $|u_0 - u_1| = d(A)$. Instead, we now have $d(A)/2 \leq |u_0 - u_1| \leq d(A)$. If $\bar{u}$ is normalized and maximal in A, then $1 \leq d(A) \leq 2$.

2.3. *Solar systems.* Let $c \geq 1$. A compact set $A \subset \mathsf{R}^n$ is said to be a c-**solar system** if there is a maximal sequence $\bar{u} = (u_0, \ldots, u_n)$ in A such that

(S1) $|u_k - u_0| \leq ch_k$ for all $2 \leq k \leq n$,

(S2) $A \setminus \{u_1, \ldots, u_n\} \subset \bar{B}(u_0, ch_n)$,

where $h_k = h_k(\bar{u})$ is as in (2.2).

The conditions (S1) and (S2) can also be expressed as the single condition

(S) $A \setminus \{u_1, \ldots, u_{k-1}\} \subset \bar{B}(u_0, ch_k)$ for all $2 \leq k \leq n$.

Observe that (S1) holds trivially for $k = 1$ with $c = 1$. If $k \geq 2$ and $u_k \neq u_0$, we can consider the angle α_k between the vector $u_k - u_0$ and the $(k-1)$-plane E_{k-1}. Condition (S1) can then be written as $\sin \alpha_k \geq 1/c$. Thus the angles α_k are bounded from below. Moreover, since $h_1 \geq \ldots \geq h_n$, (S1) implies that

$$|u_k - u_0| \leq c|u_j - u_0| \quad \text{for } 1 \leq j < k \leq n,$$

but there is no upper bound for the ratios $|u_j - u_0|/|u_{j+1} - u_0|$. Condition (S2) means that most of A is concentrated to a neighborhood of u_0, which can be arbitrarily small.

We can think that the points $u_1, \ldots, u_n$ are the **planets** and that the rest $A \setminus \{u_1, \ldots, u_n\}$ of the set A is the **sun** of the system. The sun is contained in the ball $\bar{B}(u_0, ch_n)$ but it is otherwise an arbitrary set. Compared with the real solar system, there are several differences: (1) The planets do not lie in a plane. On the contrary, the vectors $u_j - u_0$ are linearly independent in a quantitative way. (2) The last planet u_n and maybe some other planets lie in the ball $\bar{B}(u_0, ch_n)$ and hence in some sense inside the sun though not too close to the center u_0. (3) The planets have no moons.

If $\dim \operatorname{aff} A = k < n$, then $h_j = 0$ and $u_j = u_0$ for $k + 1 \le j \le n$. This means that the system **degenerates** to the finite set $A = \{u_0, \dots, u_k\}$. Hence $\dim \operatorname{aff} A = n$ whenever $\#A \ge n + 1$.

It is possible to characterize the solar systems without using maximal sequences; see 2.10.

2.4. *Examples.* 1. If $t_j > 0$ for $1 \le j \le n$, the set $A = \{0, t_1e_1, \dots, t_ne_n\}$ is a 1-solar system .

2. For $0 < t < 1$, the set $A = \{0, e_1, te_2, e_1 + te_2\}$ is not a c-solar system for any $c \le 1/t$.

3. Suppose that $A \subset \mathsf{R}^n$ is compact and that $\bar{B}(u_0, r) \subset A \subset \bar{B}(u_0, R)$. If $(u_0, \dots, u_n)$ is a maximal sequence in A, then $R \ge h_1 \ge \dots \ge h_n \ge r$. It follows that A is a c-solar system with $c = R/r$.

4. In particular, the closure of a bounded c-John domain $D \subset \mathsf{R}^n$ in the distance carrot sense [NV, 2.2] is a c-solar system .

5. If $\theta(A) \ge qd(A) > 0$, then A is a c-solar system with $c = 1/q$. To prove this, let $\bar{u}$ be a maximal sequence in A. Since $qd(A) \le \theta(A) \le h_n$, we have $A \subset \bar{B}(u_0, h_n/q)$, and the conditions (S1) and (S2) follow with $c = 1/q$.

6. Every compact set $A \subset \mathsf{R}$ is trivially a 1-solar system .

We can now formulate the main result of the paper.

2.5. THEOREM: *The properties c-IAP and c-solar system are quantitatively equivalent. More precisely, let $A \subset \mathsf{R}^n$ be compact.*

(1) *If A is a c-solar system , then A has the c^*-IAP with $c^* = c^*(c, n)$.*

(2) *If A has the c-IAP , then A is a c'-solar system with $c' = c'(c, n)$.*

We shall prove (1) in Section 4 and (2) in Section 5. In Section 3 we give some general results on the IAP.

We first give some consequences of 2.5:

2.6. THEOREM: *Suppose that $A \subset \mathsf{R}^n$ is a compact set with $\dim \operatorname{aff} A = k < n$. Then A has the c-IAP if and only if, quantitatively, $\#A = k + 1$ and A can be written as a maximal sequence $(u_0, \dots, u_k)$ such that*

$$|u_j - u_0| \le ch_j = cd(u_j, \operatorname{aff}\{u_0, \dots, u_{j-1}\})$$

whenever $2 \le j \le k$.

2.7. THEOREM: *Let $A \subset \mathsf{R}^n$ be compact with $\#A \geq n+1$. Then A has the c-IAP for some $c \geq 1$ if and only if $\theta(A) > 0$.*

Proof: If $\theta(A) > 0$, then A has the c-IAP with $c = c_n d(A)/\theta(A)$ by [ATV, 3.3]. Alternatively, this follows from 2.5 and 2.4.5.

Conversely, suppose that A has the c-IAP. Since $\#A \geq n+1$, we have $\dim \operatorname{aff} A = n$ by 2.6. Hence $\theta(A) > 0$. ∎

2.8. THEOREM: *If $A \subset \mathsf{R}^n$ is a compact set without isolated points, the following conditions are quantitatively equivalent:*

(1) *A has the c-IAP,*

(2) *$\theta(A) \geq qd(A)$.*

Proof: The implication (2) $\Rightarrow$ (1) was given in [ATV, 3.3], and it is recalled in 3.3 of the present paper. If (1) holds, then A is a c'-solar system with $c' = c'(c, n)$ by 2.5. Let $\bar{u} = (u_0, \dots, u_n)$ be the maximal sequence in A given by the definition of a solar system. Since u_1 is not isolated in A, we have

$$\sup\{|x - u_0| \colon x \in A \setminus \{u_1, \dots, u_n\}\} = |u_1 - u_0|,$$

and hence $d(A)/2 \leq |u_1 - u_0| \leq c' h_n(\bar{u})$. This implies that

$$\theta(A) \geq h_n/C_n \geq d(A)/2c'C_n,$$

where C_n depends only on n; see 5.8. ∎

The following result on simplexes will be needed in 2.10 and in 5.27.

2.9. LEMMA: *Suppose that $\Delta \subset \mathsf{R}^n$ is a p-simplex with vertices $0, u_1, \dots, u_p$ and that $\bar{u} = (0, u_1, \dots, u_p)$ is a maximal sequence in Δ. Let b_j be the height of Δ measured from the vertex u_j. Then*

$$\frac{|u_j|}{b_j} \leq \frac{|u_1| \cdots |u_p|}{h_1 \cdots h_p},$$

where $h_k = h_k(\bar{u})$ is as in (2.2).

Proof: Let Δ_j be the $(p-1)$-face of Δ opposite to u_j. Then the volume of Δ is $m_p(\Delta) = b_j m_{p-1}(\Delta_j)/p$. Here

$$m_{p-1}(\Delta_j) \leq \frac{|u_1| \cdots |u_p|}{(p-1)!|u_j|}.$$

Since $m_p(\Delta) = h_1 \cdots h_p/p!$, the lemma follows. ∎

We next give some alternative characterizations of solar systems, which do not involve maximal sequences. This result is not needed in the proof of the main theorem 2.5.

2.10. THEOREM: *Let $A \subset \mathbb{R}^n$ be compact. The following conditions are quantitatively equivalent:*

(1) *A is a c-solar system.*

(2) *There is a finite set $F = \{u_0, \ldots, u_n\} \subset A$ such that*

  (2a) $|u_k - u_0| \le c d(u_k, \operatorname{aff}(F \setminus \{u_k\})$ *for all* $1 \le k \le n$,

  (2b) $A \setminus F \subset \bar{B}(u_0, c \min\{|u_k - u_0| \colon 1 \le k \le n\})$.

(3) *There is a sequence $\bar{u} = (u_0, \ldots, u_n)$ in A such that*

  (3a) $|u_{k+1} - u_0| \le |u_k - u_0|$ *for* $1 \le k \le n-1$,

  (3b) $|u_k - u_0| \le c h_k$ *for* $1 \le k \le n$,

  (3c) $A \setminus \{u_1, \ldots, u_n\} \subset \bar{B}(u_0, c|u_n - u_0|)$.

Here $h_k = h_k(\bar{u})$ is as in (2.2).

(4) *There is a sequence $\bar{u} = (u_0, \ldots, u_n)$ in A such that*

$$A \setminus \{u_1, \ldots, u_{k-1}\} \subset \bar{B}(u_0, c h_k)$$

for all $1 \le k \le n$.

Proof: We prove the case aff $A = \mathbb{R}^n$; the degenerate case is obtained by an easy modification. By an auxiliary translation we may assume that $u_0 = 0$ in each condition. Observe that (2) is independent of the order of the points $u_1, \ldots, u_n$. We prove the quantitative implications (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (1).

(1) ⇒ (2): Let $\bar{u} = (0, u_1, \ldots, u_n)$ be a maximal sequence given by the definition 2.3 of a c-solar system. Since $|u_j| \le c h_j$, Lemma 2.9 gives (2a) with $c \mapsto c^n$. Furthermore, since $h_n \le h_j \le |u_j|$ for all $1 \le j \le n$, (2b) follows from (S2) in 2.3.

(2) ⇒ (3): By rearranging we may assume that $|u_1| \ge \cdots \ge |u_n|$. Then (3) holds with the same constant c.

(3) ⇒ (4): For $1 \le k \le j \le n$ we have $|u_j| \le |u_k| \le c h_k$. Hence (4) holds with the same constant c.

(4) ⇒ (1): We need the following result.

2.11. LEMMA: *Suppose that $\bar{a} = (0, a_1, \ldots, a_k)$ is a sequence in $\mathbb{R}^n$ such that $|a_j| = 1 \le c h_j(\bar{a})$ for $1 \le j \le k$. Suppose also that $E \subset \mathbb{R}^n$ is a linear subspace* *with* $\dim E = k-1$. *Then there is* $j \in [1,k]$ *such that* $d(a_j, E) \geq 1/\lambda$ *with* $\lambda = \lambda(c,k) \geq 1$.

Proof: Let Δ be the k-simplex with vertices $0, a_1, \ldots, a_k$, set $F = \operatorname{aff} \Delta$, and let $F \cap \bar{B}(x_0, r)$, $x_0 \in F$, be the k-disk inscribed to Δ. Since $\dim F + \dim E^{\perp} = n+1$, we can choose a unit vector $e \in F \cap E^{\perp}$. Let $P'\colon \mathsf{R}^n \to E^{\perp}$ be the orthogonal projection. Then $|P'(x_0+re) - P'(x_0-re)| = 2r$, and thus $d(P'\Delta) \geq 2r$. Hence it suffices to get an estimate $r \geq 1/\lambda$.

Let Δ_j be the $(k-1)$-face of Δ opposite to a_j. The k-volume of Δ is $m_k(\Delta) = (r/k)\sum_{j=0}^{k} m_{k-1}(\Delta_j)$. Here $m_{k-1}(\Delta_j) \leq 1/(k-1)!$ for $1 \leq j \leq k$, and $m_{k-1}(\Delta_0) \leq 2^{k-1}\alpha(k-1)$, where $\alpha(k-1) = 2^{(1-k)/2}\sqrt{k}/(k-1)!$ is the volume of the unit $(k-1)$-simplex. Since $m_k(\Delta) = h_1 \cdots h_k/k!$, we obtain $r \geq 1/\lambda$ with $\lambda = 2^{(k+1)/2}c^{k-1}\sqrt{k}$. ∎

2.12. *Proof of* 2.10 *continues:* Suppose that $\bar{u} = (u_0, \ldots, u_n)$ satisfies condition (4) of 2.10. Choose a maximal sequence $\bar{v} = (v_0, \ldots, v_n)$ in A with $v_0 = u_0$. We may assume that $\bar{v}$ is normalized. Then $u_0 = v_0 = 0$, $v_1 = e_1$ and $A \subset \bar{B}^n$.

Set $h'_k = h_k(\bar{v})$. We first show that

$$h_k \leq c\lambda(c,k)h'_k \tag{2.13}$$

for all $1 \leq k \leq n$, where $\lambda(c,k)$ is the constant of 2.11.

Applying 2.11 to $a_j = u_j/|u_j|$ we find $j \in [1,k]$ with $d(a_j, \mathsf{R}^{k-1}) \geq 1/\lambda(c,k)$. Hence $d(u_j, \mathsf{R}^{k-1}) \geq |u_j|/\lambda(c,k)$. Since $\bar{v}$ is a maximal sequence in A, we have $d(u_j, \mathsf{R}^{k-1}) \leq h'_k$, and thus $|u_j| \leq \lambda(c,k)h'_k$. Since $h_k \leq |u_k| \leq ch_j \leq c|u_j|$, we obtain (2.13).

Set $c'_1 = 1$ and $c'_{k+1} = c^2\lambda(c,k+1)c'_k$ for $1 \leq k \leq n-1$. We show that

$$A \setminus \{v_1, \ldots, v_{k-1}\} \subset \bar{B}(c'_k h'_k) \tag{2.14}$$

for all $1 \leq k \leq n$. This will prove that A is a c-solar system with $c' = c'_n$.

The case $k = 1$ is clear, since $A \subset \bar{B}^n = \bar{B}(c'_1 h'_1)$. Assume that (2.14) holds for $1 \leq k \leq p \leq n-1$. Let $x \in A \setminus \{v_1, \ldots, v_p\}$.

If $\{u_1, \ldots, u_p\} = \{v_1, \ldots, v_p\}$, then $|x| \leq ch_{p+1}$, and hence $|x| \leq c^2\lambda(c,p+1)h'_{p+1} \leq c'_{p+1}h'_{p+1}$ by (2.13).

If $\{u_1, \ldots, u_p\} \neq \{v_1, \ldots, v_p\}$, then we can choose $j \in [1,p]$ with $v_j \notin \{u_1, \ldots, u_p\}$. Then $|v_j| \leq ch_{p+1} \leq c^2\lambda(c,p+1)h'_{p+1}$ by (2.13). Since $x \in A \setminus \{v_1, \ldots, v_{j-1}\}$, the inductive hypothesis yields

$$|x| \leq c'_j h'_j \leq c'_j |v_j| \leq c'_j c^2 \lambda(c,p+1)h'_{p+1} \leq c'_{p+1}h'_{p+1}. \qquad ∎$$

3. General results on the IAP

We first show that the c-IAP is invariant under similarities.

3.1. THEOREM: *Suppose that $A \subset \mathsf{R}^n$ has the c-IAP and that $T: \mathsf{R}^n \to \mathsf{R}^n$ is a similarity. Then the set TA has the c-IAP .*

Proof: Let L be the Lipschitz constant of T, and let $f: TA \to \mathsf{R}^n$ be an ε-nearisometry . Define $g: A \to \mathsf{R}^n$ by $gx = T^{-1}fTx$. Then g is an (ε/L)-nearisometry . Hence there is an isometry S of R^n with $\|S - g\|_A \le c\varepsilon/L$. Then $S' = TST^{-1}$ is an isometry with $\|S - f\|_{TA} \le c\varepsilon$. ∎

3.2. THEOREM: *Suppose that $A \subset \mathsf{R}^n$ is compact, that $\varepsilon_0 > 0$, and that A satisfies the definition of c-IAP for all $\varepsilon < \varepsilon_0$. Then A has the c'-IAP with $c' = \max\{c, 1 + 2d(A)/\varepsilon_0\}$.*

Proof: Let $\varepsilon \ge \varepsilon_0$ and let $f: A \to \mathsf{R}^n$ be an ε-nearisometry . Fix $a \in A$, and let $S: A \to \mathsf{R}^n$ be the isometry defined by $Sx = x + fa - a$. For each $x \in A$ we have

$$\begin{aligned}|Sx - fx| &\le |x - a| + |fx - fa| \le 2|x - a| + \varepsilon \\ &\le 2|x - a|\varepsilon/\varepsilon_0 + \varepsilon \le (1 + 2d(A)/\varepsilon_0)\varepsilon.\end{aligned}$$ ∎

3.3. THEOREM: *Suppose that $A \subset \mathsf{R}^n$ is compact and that $\theta(A) \ge qd(A) > 0$. Then A has the c-IAP with $c = c_n/q$, where c_n depends only on n.*

Proof: The theorem follows from [ATV, 3.3]. ∎

4. Solar systems have the IAP

In this section we prove Part (1) of the main theorem 2.5.

4.1. THEOREM: *If $A \subset \mathsf{R}^n$ is a c-solar system , then A has the c^*-IAP with $c^* = c^*(c, n)$.*

The proof of 4.1 is rather similar to but (surprisingly) somewhat simpler than the proof of [ATV, 3.3]. On the other hand, the function $c \mapsto c^*$ will not be so simple as the function $q \mapsto c_n/q$ of [ATV, 3.3].

If $\bar{u} = (0, u_1, \ldots, u_m)$ is a normalized maximal sequence in a compact set $F \subset \mathsf{R}^n$, we say that a map $f: F \to \mathsf{R}^n$ is **normalized** at $\bar{u}$ if $f(0) = 0$ and if $f(u_k) \in \mathsf{R}^k_+$ for all $1 \le k \le m$.

Convention. In this section we shall write $x' = fx$ if f is a map defined at a point x.

4.2. *An inductive statement.* For an integer $n \geq 1$ we consider the following statement.

T_n: Suppose that $n \leq N$ and that $F = \{u_0, \ldots, u_n, x\} \subset \mathsf{R}^N$ is such that the sequence $\bar{u} = (u_0, \ldots, u_n)$ normalized and maximal in F with $h_n > 0$. Suppose also that

$$\begin{aligned} |u_k| &\leq c h_k \quad \text{for } 1 \leq k \leq n, \\ |x| &\leq c h_n, \end{aligned}$$

where h_k is as in (2.2). Let $f\colon F \to \mathsf{R}^N$ be an ε-nearisometry with $\varepsilon \leq h_n$, normalized at $\bar{u}$. Then

(i) $|x_n - x'_n| \leq \varrho_n \varepsilon$,

(ii) $|x^2_{(n+1)*} - x'^2_{(n+1)*}| \leq \tau_n(|x| \vee \varepsilon)\varepsilon$.

The constants ϱ_n and τ_n depend only on c and n, and they are given by the formulas

$$\begin{aligned} \varrho_1 &= 7.5, \ \tau_1 = 25.5, \\ \varrho_n &= 3c\left(2 + \sum_{k=1}^{n-1} \varrho_k\right) + 2c^2 \tau_{n-1}, \\ \tau_n &= \tau_{n-1} + 3\varrho_n. \end{aligned}$$

4.3. Lemma: *Statement T_n is true for all $n \geq 1$.*

Proof: We use induction on n. Let first $n = 1$. We have $u_0 = 0$, $u_1 = e_1$, $f(0) = 0$ and $fe_1 = \alpha e_1$ with $|\alpha - 1| \leq \varepsilon$. Estimating the number $|x_1 - x'_1| = |x \cdot e_1 - x' \cdot e_1|$ by the basic formula

$$2a \cdot b = |a|^2 + |b|^2 - |a - b|^2 \tag{4.4}$$

we get

$$2|x_1 - x'_1| \leq (|x| + |x'|)\big||x| - |x'|\big| + (|x - e_1| + |x' - e_1|)\big||x - e_1| - |x' - e_1|\big|.$$

Here

$$\begin{aligned} &|x| \leq 1, \ |x'| \leq 1 + \varepsilon, \ |x - e_1| \leq 2, \\ &|x' - e_1| \leq |x' - \alpha e_1| + |\alpha - 1| \leq |x - e_1| + 2\varepsilon, \\ &|x' - e_1| \geq |x' - \alpha e_1| - |\alpha - 1| \geq |x - e_1| - 2\varepsilon, \end{aligned}$$

and we obtain

$$2|x_1 - x'_1| \leq (2 + \varepsilon)\varepsilon + (4 + 2\varepsilon)2\varepsilon = 10\varepsilon + 5\varepsilon^2 \leq 15\varepsilon,$$

since $\varepsilon \leq h_1 = 1$. Hence T_1(i) holds with $\varrho_1 = 7.5$.

To obtain T_1(ii) we use the formula $x_{2*}^2 = |x|^2 - x_1^2$ and get

$$|x_{2*}^2 - {x'}_{2*}^2| \le (|x| + |x'|)\big||x| - |x'|\big| + (|x_1| + |x'_1|)|x_1 - x'_1|.$$

Here

$$|x'| \le |x| + \varepsilon \le 2(|x| \vee \varepsilon), \ |x_1| \le |x|, \ |x'_1| \le |x'|, \ |x_1 - x'_1| \le 7.5\varepsilon,$$

and hence

$$|x_{2*}^2 - {x'}_{2*}^2| \le 3(|x| \vee \varepsilon)\varepsilon + 3(|x| \vee \varepsilon)7.5\varepsilon = 25.5(|x| \vee \varepsilon)\varepsilon.$$

This proves T_1(ii) with $\tau_1 = 25.5$.

Next assume that $n \ge 2$ and that T_k is true for $1 \le k \le n-1$. Let $f\colon F \to \mathsf{R}^N$ be as in T_n. By (4.4) we obtain

$$\begin{aligned} 2|x \cdot u_n - x' \cdot u'_n| \le &(|x| + |x'|)\big||x| - |x'|\big| + (|u_n| + |u'_n|)\big||u_n| - |u'_n|\big| \\ &+ (|x - u_n| + |x' - u'_n|)\big||x - u_n| - |x' - u'_n|\big|. \end{aligned}$$

Here

$$\begin{aligned} &|x| \le ch_n, \ |u_n| \le ch_n, \\ &|x'| \le |x| + \varepsilon \le ch_n + h_n \le 2ch_n, \\ &|u'_n| \le |u_n| + \varepsilon \le 2ch_n, \\ &|x' - u'_n| \le |x - u_n| + \varepsilon \le 3ch_n, \end{aligned}$$

and hence

$$2|x \cdot u_n - x' \cdot u'_n| \le 3ch_n\varepsilon + 3ch_n\varepsilon + 5ch_n\varepsilon = 11ch_n\varepsilon.$$

By normalization we have $u_n \in \mathsf{R}^n$, $h_n = (u_n)_n$ and $u'_n \in \mathsf{R}^n$. Setting $h'_n = (u'_n)_n$ we get

$$|x_n h_n - x'_n h'_n| \le 6ch_n\varepsilon + \sum_{k=1}^{n-1} \big(|x_k - x'_k||(u_n)_k| + |x'_k||(u_n)_k - (u'_n)_k|\big). \tag{4.5}$$

To estimate the terms of the sum, we use condition T_k for the maps $f|A_k$ and $f|B_k$, where

$$A_k = \{u_0, \ldots, u_k, x\}, \quad B_k = \{u_0, \ldots, u_k, u_n\}.$$

The conditions $|x| \le ch_k$, $|u_n| \le ch_k$, $\varepsilon \le h_k$ hold, since $h_k \ge h_n$. Applying T_k(i) to $f|A_k$ and to $f|B_k$ we get

$$|x_k - x'_k| \le \varrho_k\varepsilon, \ |(u_n)_k - (u'_n)_k| \le \varrho_k\varepsilon.$$

Since

$$|(u_n)_k| \le |u_n| \le ch_n,$$
$$|x'_k| \le |x'| \le |x| + \varepsilon \le ch_n + h_n \le 2ch_n,$$

these estimates and (4.5) yield

$$|x_n h_n - x'_n h'_n| \le 3ch_n\varepsilon\Big(2 + \sum_{k=1}^{n-1} \varrho_k\Big). \tag{4.6}$$

Applying T_{n-1}(ii) to $f|B_{n-1}$ and observing that $h_n = (u_n)_{n*}$, $h'_n = (u'_n)_{n*}$ we obtain

$$|h_n^2 - {h'}_n^2| \le \tau_{n-1}(|u_n| \vee \varepsilon)\varepsilon.$$

Since $\varepsilon \le h_n$ and $|u_n| \le ch_n$, this yields

$$|h_n - h'_n| \le |h_n^2 - {h'}_n^2|/h_n \le c\tau_{n-1}\varepsilon.$$

Since $|x'_n| \le 2ch_n$, this and (4.6) imply that

$$\begin{aligned} h_n|x_n - x'_n| &\le |x_n h_n - x'_n h'_n| + |x'_n||h_n - h'_n| \\ &\le 3ch_n\varepsilon\Big(2 + \sum_{k=1}^{n-1} \varrho_k\Big) + 2c^2 h_n \tau_{n-1}\varepsilon = h_n \varrho_n \varepsilon, \end{aligned}$$

and hence T_n(i) is true.

Since $|x_n| + |x'_n| \le |x| + |x| + \varepsilon \le 3(|x| \vee \varepsilon)$, conditions T_n(i) and T_{n-1}(ii) imply T_n(ii):

$$\begin{aligned} |x_{(n+1)*}^2 - {x'}_{(n+1)*}^2| &\le |x_{n*}^2 - {x'}_{n*}^2| + |x_n^2 - {x'}_n^2| \\ &\le \tau_{n-1}(|x| \vee \varepsilon)\varepsilon + (|x_n| + |x'_n|)\varrho_n\varepsilon \\ &= \tau_n(|x| \vee \varepsilon)\varepsilon. \qquad \blacksquare \end{aligned}$$

Proof of 4.1: Since the c-solar system condition and the c-IAP are invariant under similarities of R^n, we may assume that A has a normalized maximal sequence $\bar{u} = (u_0, \dots, u_n)$ satisfying the conditions

(S1) $|u_k| \le ch_k$ for all $2 \le k \le n$,

(S2) $A \setminus \{u_1, \dots, u_n\} \subset \bar{B}(ch_n)$,

where now $h_k = (u_k)_k$.

It is possible that $\dim \operatorname{aff} A = m < n$. In this case we have $u_k = 0$ and $h_k = 0$ for $m + 1 \le k \le n$, and $A = \{u_0, \dots, u_m\}$.

Suppose that $f\colon A \to \mathsf{R}^n$ is an ε-nearisometry . We may assume that f is normalized at $\bar{u}$. It suffices to show that

$$|x - x'| \le c^*\varepsilon \tag{4.7}$$

for all $x \in A$ with $c^* = c^*(c, n)$.

We use induction on n and start with the case $n = 1$. Now $A \subset \mathsf{R}$ and $f\colon A \to \mathsf{R}$ is an ε-nearisometry with $f(0) = 0$, $f(1) = \alpha$, $|\alpha - 1| \le \varepsilon$. Let $x \in A$. Since

$$\alpha - x' \le |\alpha - x'| \le |1 - x| + \varepsilon = 1 - x + \varepsilon,$$

we have

$$x - x' \le 1 - \alpha + \varepsilon \le 2\varepsilon.$$

If $x \ge 0$, then

$$x' \le |x'| \le |x| + \varepsilon = x + \varepsilon,$$

and hence $x' - x \le \varepsilon$. Assume that $x < 0$. If $\varepsilon \ge 2/3$, then

$$|x - x'| \le |x| + |x'| \le 2|x| + \varepsilon \le 2 + \varepsilon \le 4\varepsilon.$$

If $\varepsilon < 2/3$, we first show that $x' < \alpha$. Assuming $x' \ge \alpha$ we get

$$x' - \alpha = |x' - \alpha| \ge |x - 1| - \varepsilon = 1 - x - \varepsilon,$$
$$x' \le |x'| \le |x| + \varepsilon = -x + \varepsilon,$$

and we obtain the contradiction

$$1 \le 2\varepsilon - \alpha \le 2\varepsilon - (1 - \varepsilon) < 1.$$

Since $x' < \alpha$, we get

$$1 + \varepsilon - x' \ge \alpha - x' = |\alpha - x'| \ge 1 - x - \varepsilon,$$

and hence $x' - x \le 2\varepsilon$. The case $n = 1$ is now proved with $c^*(c, 1) = 4$.

It is natural that $c^*(c, 1)$ does not depend on c, because every compact set in R is a 1-solar system.

Next let $n \ge 2$, and assume that the theorem holds in dimensions $m \le n - 1$.

If $h_n = 0$, then $A = \{u_0, \ldots, u_m\}$ for some $m < n$ with $h_m > 0$. Since f is normalized at $\bar{u}$, we have $fA \subset \mathsf{R}^m$. Hence $|x - x'| \le c^*(c, m)\varepsilon$ for all $x \in A$ by the inductive hypothesis.

Assume that $h_n > 0$. Let $x \in A$. If $x \in A_0 = \{u_0, \ldots, u_{n-1}\}$, then $|x - x'| \le c^*(c, n - 1)\varepsilon$. Assume that $x \in A \setminus A_0$. Then $|x| \le ch_n$ by (S1) and (S2).

If $\varepsilon \le h_n$, we apply statement T_k(i) to $f|\{u_0, \ldots, u_k, x\}$ for each $k \in [1, n]$ and get $|x_k - x'_k| \le \varrho_k \varepsilon$. Hence

$$|x - x'|^2 \le \varepsilon^2 \sum_{k=1}^{n} \varrho_k^2,$$

which gives (4.7) with $c^*(c,n) = \left(\sum_{k=1}^n \varrho_k^2\right)^{1/2}$.

Finally, if $\varepsilon > h_n$, then

$$|x - x'| \leq |x| + |x'| \leq 2|x| + \varepsilon \leq 2ch_n + \varepsilon \leq (2c+1)\varepsilon. \qquad \blacksquare$$

5. Sets with the IAP are solar systems

In this section we prove Part (2) of the main theorem 2.5.

5.1. THEOREM: *If a compact set $A \subset \mathsf{R}^n$ has the c-IAP , then A is a c'-solar system with $c' = c'(c,n)$.*

We first consider some auxiliary maps needed in the proof of 5.1. Let $n \geq 2$ and let $\omega\colon \mathsf{R} \to \mathsf{R}$ be continuous. We define a homeomorphism $g_\omega\colon \mathsf{R}^n \to \mathsf{R}^n$ by

$$g_\omega(x) = (x_1, \dots, x_{n-1}, x_n + \omega(x_1)). \tag{5.2}$$

5.3. LEMMA: *For all $x, y \in \mathsf{R}^n$ with $x_1 \neq y_1$ we have*

$$\big||g_\omega x - g_\omega y| - |x - y|\big| \leq \frac{|x_n - y_n||\omega(x_1) - \omega(y_1)|}{|x_1 - y_1|} + \frac{(\omega(x_1) - \omega(y_1))^2}{2|x_1 - y_1|}.$$

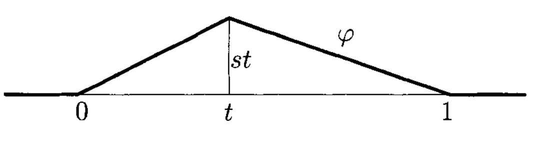
Proof: Let $\pi\colon \mathsf{R}^n \to \mathsf{R}^{n-1}$ be the orthogonal projection. Then

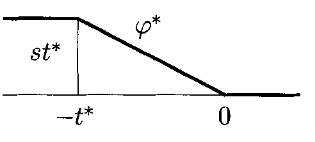
$$\begin{aligned}|g_\omega x - g_\omega y|^2 &= |\pi x - \pi y|^2 + (x_n + \omega(x_1) - y_n - \omega(y_1))^2 \\ &= |x - y|^2 + 2(x_n - y_n)(\omega(x_1) - \omega(y_1)) + (\omega(x_1) - \omega(y_1))^2.\end{aligned}$$

Since $|g_\omega x - g_\omega y| + |x - y| \geq 2|x_1 - y_1|$, the lemma follows. $\blacksquare$

5.4. *The functions φ and φ^*.* For $0 < t \leq 1/2$ and $s > 0$ we let $\varphi = \varphi_{st}\colon \mathsf{R} \to \mathsf{R}$ be the piecewise linear function such that $\varphi(0) = \varphi(1) = 0$, $\varphi(t) = st$, $\varphi(r) = 0$ for $r \notin [0,1]$, and φ is affine on the intervals $[0,t]$ and $[t,1]$. For $t^* > 0$ we also define a piecewise linear function $\varphi^* => \varphi^*_{s,t^*}\colon \mathsf{R} \to \mathsf{R}$ by

$$\varphi^*(r) = \begin{cases} st^* & \text{for } r \leq -t^*, \\ -sr & \text{for } -t^* \leq r \leq 0, \\ 0 & \text{for } r \geq 0. \end{cases}$$



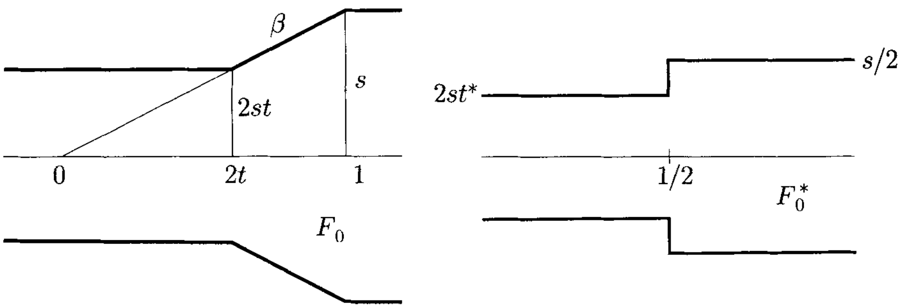


Next define $\beta = \beta_{st}\colon \mathsf{R} \to \mathsf{R}$ by

$$\beta(r) = \begin{cases} 2st & \text{for } r \le 2t, \\ sr & \text{for } 2t \le r \le 1, \\ s & \text{for } r \ge 1, \end{cases}$$

and set

$$F_0 = F_0(s,t) = \{x \in \mathsf{R}^n \colon |x_n| \le \beta(x_1)\},$$
$$F_0^* = F_0^*(s,t^*) = \{x \in \mathsf{R}^n \colon x_1 \le 1/2,\ |x_n| \le 2st^*\}$$
$$\cup \{x \in \mathsf{R}^n \colon x_1 \ge 1/2,\ |x_n| \le s/2\}.$$



5.5. Lemma: *The map $g_\varphi | F_0$ is an ε_0-nearisometry with $\varepsilon_0 = 5s^2t$.*

Proof: Let $x, y \in F_0$ and set $\delta = \big||g_\varphi x - g_\varphi y| - |x-y|\big|$. We may assume that $x_1 < y_1$.

Case 1: $y_1 \le 2t$. Now

$$|x_n - y_n| \le 4st, \ |\varphi(x_1) - \varphi(y_1)| \le s|x_1 - y_1| \wedge st,$$

and 5.3 gives $\delta \le 4.5s^2t < \varepsilon_0$.

Case 2: $x_1 \ge t$. Now

$$|x_n - y_n| \le 2s, \ |\varphi(x_1) - \varphi(y_1)| \le 2st|x_1 - y_1| \wedge st,$$

and 5.3 again gives $\delta < \varepsilon_0$.

Case 3: $x_1 \le t$, $y_1 \ge 2t$. Since

$$|x_1 - y_1| \ge y_1/2, \ |x_n - y_n| \le 2sy_1, \ |\varphi(x_1) - \varphi(y_1)| \le st,$$

we obtain $\delta < \varepsilon_0$ by 5.3. ∎

5.6. LEMMA: *The map $g_{\varphi^*}|F_0^*$ is an ε_0^*-nearisometry with $\varepsilon_0^* = 6s^2t^*(t^* \vee 1)$.*

Proof: Let $x, y \in F_0^*$, set $\delta = \big||g_{\varphi^*}x - g_{\varphi^*}y| - |x-y|\big|$ and assume that $x_1 < y_1$. If $x_1 \geq 0$, then $\delta = 0$. If $y_1 \leq 1/2$, then

$$|x_n - y_n| \leq 4st^*, \; |\varphi^*(x_1) - \varphi^*(y_1)| \leq s|x_1 - y_1| \wedge st^*,$$

and 5.3 gives $\delta \leq 4.5s^2t^* < \varepsilon_0$. If $x_1 \leq 0$ and $y_1 \geq 1/2$, then

$$|x_1 - y_1| \geq 1/2, \; |\varphi^*(x_1) - \varphi^*(y_1)| \leq st^*,$$
$$|x_n - y_n| \leq 2st^* + s/2 \leq 5s(t^* \vee 1)/2,$$

and the inequality $\delta \leq \varepsilon_0^*$ again follows from 5.3. ∎

In the proof of 5.1 we shall make use of the maps g_φ and g_{φ^*} conjugated by a similarity $T\colon \mathsf{R}^n \to \mathsf{R}^n$. Set $h_\omega = T^{-1}g_\omega T$, and let $\lambda = \operatorname{Lip} T$ be the Lipschitz constant of T. The following result is a corollary of 5.5 and 5.6.

5.7. LEMMA: *The map $h_\varphi|T^{-1}F_0$ is an ε-nearisometry with $\varepsilon = \varepsilon_0/\lambda = 5s^2t/\lambda$, and $h_{\varphi^*}|T^{-1}F_0^*$ is an ε^*-nearisometry with $\varepsilon^* = \varepsilon_0^*/\lambda = 6s^2t^*(t^* \vee 1)/\lambda$.*

We need the following result on simplexes. Recall that $\theta(X)$ is the thickness of a compact set $X \in \mathsf{R}^n$, defined in 1.1.

5.8. LEMMA: *Let $\Delta \subset \mathsf{R}^k$ be a k-simplex with vertices $u_0, \ldots, u_k$ such that $(u_0, \ldots, u_k)$ is a maximal sequence in Δ. Then $h_k \leq C_k\theta(\Delta)$, where h_k is given by (2.2), and the constant C_k depends only on k.*

Proof: This follows from [ATV, 5.3 and 5.7]. However, the proof of [ATV, 5.7] must be slightly modified, in view of the new definition of a maximal sequence. ∎

5.9. *Two special cases.* The proof of Theorem 5.1 is elementary but rather long. To follow the idea, it might be helpful for the reader to keep the following two special cases in mind. However, they are not actually needed in the proof. Let $n = 2$.

1. Assume that $\{0, e_1\} \subset A \subset [0, e_1]$ and that $\#A \geq 3$. Then A is not a c-solar system in R^2 for any c. To show that A does not have the IAP we may assume that $a = te_1 \in A$ with $0 < t \leq 1/2$. Let $0 < s < 1$ and consider the map $g\colon A \to \mathsf{R}^2$ defined by $g(re_1) = re_1 + \varphi(r)e_2$, where $\varphi = \varphi_{st}\colon \mathsf{R} \to \mathsf{R}$ is defined in 5.4. By 5.5, the map g is an ε-nearisometry with $\varepsilon = 5s^2t$.

If A has the c-IAP, there is an isometry $S\colon \mathsf{R}^2 \to \mathsf{R}^2$ with $\|S-g\|_A \le c\varepsilon$. Then $S\mathsf{R}$ is a line meeting the disks $\bar{B}(y, c\varepsilon)$ for $y = 0, e_1, ga$. Since $ga = te_1 + ste_2$, this implies that $c\varepsilon \ge st/2$, and hence $1 \le 10cs$. As $s \to 0$, this gives a contradiction.

An elaboration of this proof shows that if $h < 1/10c$ and if $\{0, e_1\} \subset A \subset [0,1] \times [-h, h]$, then A contains no point x with $5ch < x_1 < 1 - 5ch$.

2. Let $0 < t < 1$ and let $A = \{0, e_1, te_2, e_1 + te_2\} \subset \mathsf{R}^2$. The set A is not a c-solar system for $c \le 1/t$. We show that if A has the c-IAP, then $c \ge 1/5t$.

Now we cannot make use of a map of the type g_ω as in Example 1. Instead, we define a map $f\colon A \to \mathsf{R}^2$ by $f(te_2) = -te_2$ and by $fx = x$ for $x \ne te_2$. Then f is an ε-nearisometry with $\varepsilon = 2t^2$.

If A has the c-IAP, there is an isometry $S\colon \mathsf{R}^2 \to \mathsf{R}^2$ with $\|S - f\|_A \le c\varepsilon$. Setting $Tx = Sx - S(0)$ we get an orthogonal map $T\colon \mathsf{R}^2 \to \mathsf{R}^2$ with $\|T - f\|_A \le 2c\varepsilon$. Since $|Te_1 - e_1| \le 2c\varepsilon$, there is an orthogonal map T_1 such that $T_1Te_1 = e_1$ and $|T_1 - I| \le 2c\varepsilon$, where $I = \mathrm{id}$ and $|T_1 - I|$ is the operator norm. Then $U = T_1T$ is an orthogonal map with $U|\mathsf{R} = \mathrm{id}$ and $|U - T| = |T_1 - I| \le 2c\varepsilon$. Then either $U = I$ or U is the reflection $Ux = (x_1, -x_2)$. In the first case we have $|U(te_1) - f(te_1)| = 2t$, in the second case $|U(e_1 + te_2) - f(e_1 + te_2)| = 2t$. On the other hand,

$$\|U - f\|_A \le |U - T| d(A) + \|T - f\|_A \le 2c\varepsilon d(A) + 2c\varepsilon.$$

Since $d(A) < 3/2$, we obtain $2t \le 10ct^2$, and hence $c \ge 1/5t$.

5.10. *Proof of* 5.1 *begins.* Choose a point $u_0 \in A$ such that $d(u_0, A \setminus \{u_0\})$ is minimal. Thus u_0 is a cluster point of A if A is an infinite set. Let $\bar{u} = (u_0, \ldots, u_n)$ be a maximal sequence in A. By an auxiliary similarity we may assume that $\bar{u}$ is normalized.

We show by induction that for each integer $k \in [1, n]$, the following condition holds:

$$(P_k) \qquad A \setminus \{u_1, \ldots, u_{k-1}\} \subset \bar{B}(c'_k h_k) \quad \text{for some } c'_k = c'_k(c).$$

This will prove Theorem 5.1.

Condition (P_1) holds with $c'_1 = 1$, since $A \subset \bar{B}^n = \bar{B}(h_1)$. Assume that $1 \le k \le n-1$ and that (P_j) holds for $1 \le j \le k$. In the rest of this section we prove that (P_{k+1}) is true. This is done in a sequence of lemmas.

We first introduce some notation. Set

$$q = \frac{1}{2^{2k+6}c}, \quad M = 1 + \frac{kc'_1 \cdots c'_k}{4q}. \tag{5.11}$$

We define a number $\mu > 0$ by $\mu = 1/k$ if A is infinite and by

$$\mu = \frac{1}{k} \wedge \frac{C_k}{3Mq} \tag{5.12}$$

if A is finite, where C_k is given by 5.8. Moreover, we set $\alpha = \mu q/C_k$. The numbers q, M, μ, α depend only on c and k.

If $h_{k+1} \geq \alpha h_k$, then

$$A \setminus \{u_1, \ldots, u_k\} \subset A \setminus \{u_1, \ldots, u_{k-1}\} \subset \bar{B}(c'_k h_k) \subset \bar{B}(c'_k h_{k+1}/\alpha).$$

Hence (P_{k+1}) holds with $c'_{k+1} = c'_k/\alpha$. In the rest of this section we assume that

$$h_{k+1} < \alpha h_k. \tag{5.13}$$

Observe that this implies that $h_k > 0$. In 5.28 we shall show that (P_{k+1}) holds with $c'_{k+1} = M$.

We let $\Delta \subset \mathsf{R}^k$ denote the k-simplex with vertices $u_0, \ldots, u_k$. Then Δ is contained in the k-interval

$$Q = [-h_1, h_1] \times \cdots \times [-h_k, h_k].$$

Let $P\colon \mathsf{R}^n \to \mathsf{R}^k$ and $P'\colon \mathsf{R}^n \to \mathsf{R}^{k\perp}$ be the orthogonal projections. Then $PA \subset Q$ and $|P'x| = x_{(k+1)*}$ for all $x \in \mathsf{R}^n$.

We let $\xi_j(x)$, $0 \leq j \leq k$, denote the barycentric coordinates of a point $x \in \mathsf{R}^k$ with respect to $(u_0, \ldots, u_k)$. We extend the function ξ_j to R^n by $\xi_j(x) = \xi_j(Px)$. For each $x \in \mathsf{R}^n$ we can write

$$x = \sum_{j=0}^{k} \xi_j(x) u_j + |P'x| e,$$

where $e = e(x, k)$ is a unit vector in $\mathsf{R}^{k\perp}$.

5.14. Lemma: *For each $x \in Q$ we have*

(1) $|\xi_j(x)| \leq 2^{k-j}$ *for* $0 \leq j \leq k$,

(2) $\sum_{j=0}^{k} |\xi_j(x)| \leq 2^{k+1} - 1$.

Proof: Clearly (2) follows from (1). Let $T\colon \mathsf{R}^k \to \mathsf{R}^k$ be the linear map for which $Te_j = e_j/h_j$ for $1 \leq j \leq k$, and set $v_j = Tu_j$. The numbers $\xi_j(x)$ are the barycentric coordinates of $y = Tx$ with respect to $(v_0, \ldots, v_k)$. Now $y \in [-1, 1]^k$, $v_0 = 0$, $v_1 = e_1$, and

$$v_i = t_{i1} e_1 + \cdots + t_{i,i-1} e_{i-1} + e_i$$

for $2 \le i \le k$, where $|t_{ij}| \le 1$. Computing the coordinates y_j we obtain

$$|\xi_k(x)| = |y_k| \le 1, \quad |\xi_{k-1}(x) + \xi_k(x)t_{k,k-1}| = |y_{k-1}| \le 1,$$

and hence

$$|\xi_{k-1}(x)| \le 1 + |\xi_k(x)||t_{k,k-1}| \le 2.$$

Proceeding inductively we obtain (1). ∎

We introduce more notation. Set

$$J_k = \{0, \dots, k\}, \quad \mathcal{J}_k = \{J \subset J_k : \varnothing \ne J \ne J_k\}.$$

For $J \in \mathcal{J}_k$ we write $J' = J_k \setminus J$. For $x \in \mathsf{R}^n$ we set

$$\xi_J(x) = \sum_{j \in J} \xi_j(x).$$

Then $\xi_{J'}(x) = 1 - \xi_J(x)$. Furthermore, set

$$L_J = \operatorname{aff}\{u_j : j \in J\} \subset \mathsf{R}^k, \quad b_J = d(L_J, L_{J'}).$$

Then $b_J = b_{J'}$. If $j \in J_k$, then $b_j = b_{\{j\}}$ is the height of Δ measured from the vertex u_j, as in 2.9. Let $a_J \in L_J$ and $a_{J'} \in L_{J'}$ be points with $|a_J - a_{J'}| = b_J$. Then the vector $a_J - a_{J'}$ is perpendicular to L_J and to $L_{J'}$. Since the orthogonal projection of Δ onto the line through a_J and $a_{J'}$ is the line segment $[a_J, a_{J'}]$, we have

$$\theta(\Delta) \le |a_J - a_{J'}| = b_J. \tag{5.15}$$

By (5.13) and 5.8, this implies that

$$h_{k+1} < \mu q b_J \tag{5.16}$$

for all $J \in \mathcal{J}_k$.

5.17. Lemma: *Let $J \in \mathcal{J}_k$ and $z \in \mathsf{R}^n$. Then there is a similarity $T: \mathsf{R}^n \to \mathsf{R}^n$ such that*

(1) $T\mathsf{R}^k = \mathsf{R}^k$,
(2) $(Tx)_1 = \xi_J(x)$ *for all* $x \in \mathsf{R}^n$,
(3) $P'Tz = |P'z|e_n/b_J$,
(4) $\operatorname{Lip} T = 1/b_J$.

Proof: By the auxiliary map $x \mapsto x/b_J$ we can temporarily normalize the situation so that $b_J = 1$. Set $a = a_J$, $a' = a_{J'}$. Then $|a - a'| = 1$. Setting $Sx = x - a'$

we obtain an isometry $S: \mathsf{R}^n \to \mathsf{R}^n$ with $|Sa| = 1$. Choose orthogonal maps $U_1: \mathsf{R}^k \to \mathsf{R}^k$ and $U_2: \mathsf{R}^{k\perp} \to \mathsf{R}^{k\perp}$ such that $U_1(Sa) = e_1$, $U_2P'z = |P'z|e_n$. Then $U = U_1P + U_2P'$ is orthogonal. We show that $T = US$ is the desired similarity (now isometry).

The conditions (1) and (4) are clear. To verify (2), observe that L_J and $L_{J'}$ are perpendicular to $a - a' = Sa$, and hence TL_J and $TL_{J'}$ are perpendicular to $Ta = e_1$. Since $Ta' = 0$ and $Ta = e_1$, it follows that

$$(Tu_j)_1 = 1 \text{ for } j \in J, \quad (Tu_j)_1 = 0 \text{ for } j \in J'.$$

The maps $y \mapsto (Ty)_1$ and ξ_J agree in R^k, since they are affine and agree in the vertices of Δ. Let $x \in \mathsf{R}^n$. Since $TP = T - UP'$ and since $UP'x \in \mathsf{R}^{k\perp}$, we obtain

$$\xi_J(x) = \xi_J(Px) = (TPx)_1 = (Tx - UP'x)_1 = (Tx)_1,$$

and (2) is proved.

Since $Tz = U_1Pz + U_2P'z - Ua'$, we have $P'Tz = U_2P'z = |P'z|e_n$, and (3) follows. ∎

Unfortunately, we still must introduce some notation. For $J \in \mathcal{J}_k$ we set

$$A_J = \{x \in A: \xi_J(x) \le 1/2\}, \quad A'_J = \{x \in A: \xi_J(x) \ge 1/2\},$$
$$t_J = \max\{\xi_J(x): x \in A_J\}, \quad t^*_J = -\min\{\xi_J(x): x \in A_J\}.$$

Then

$$A'_J = A_{J'}, \quad A = A_J \cup A'_J.$$

We shall show in 5.20 that A_J and A'_J are disjoint, and hence $A'_J = A \setminus A_J$. For all $j \in J$ we have $\xi_J(u_j) = 1$, and hence $u_j \in A'_J$. Similarly $u_j \in A_J$ for $j \in J'$. Hence the sets A_J and A'_J are never empty. By 5.14 we always have

$$0 \le t_J \le 1/2, \quad 0 \le t^*_J \le 2^{k+1} - 1.$$

5.18. LEMMA: *For each $J \in \mathcal{J}_k$ there is $y \in A_J$ such that*

$$|P'y| \ge 4qb_J(t_J \vee t^*_J).$$

Proof: Assume that the lemma is not true. Then $t_J \vee t^*_J > 0$. The proof can be regarded as an elaboration of the special case 5.9.1.

CASE 1: $t^*_J \le t_J$. Now $t_J > 0$ and $|P'x| < 4qb_Jt_J$ for all $x \in A_J$. Pick $z \in A_J$ with $\xi_J(z) = t_J$. Set $t = t_J$, $s = 2q$, and let $\varphi = \varphi_{st}: \mathsf{R} \to \mathsf{R}$ be the function defined in 5.4. Let $T: \mathsf{R}^n \to \mathsf{R}^n$ be the similarity given by 5.17 for these J

and z. Let $g = g_\varphi\colon \mathsf{R}^n \to \mathsf{R}^n$ be the homeomorphism defined in (5.2), and set $h = T^{-1}gT$. By 5.7, the map $h|T^{-1}F_0$ is an ε-nearisometry with $\varepsilon = 5s^2tb_J$, where F_0 is defined in 5.4.

We show that $TA \subset F_0$. Let $x \in A$.

SUBCASE 1a: $x \in A_J$. Now $(Tx)_1 = \xi_J(x) \le t$. Since $T\mathsf{R}^k = \mathsf{R}^k$ and since $\operatorname{Lip} T = 1/b_J$, we have

$$|(Tx)_n| \le |P'Tx| = |P'x|/b_J < 4qt = 2st.$$

Hence $Tx \in F_0$.

SUBCASE 1b: $x \in A'_J$. Now $(Tx)_1 = \xi_J(x) \ge 1/2$. By (5.16) we obtain

$$|(Tx)_n| \le |P'x|/b_J \le h_{k+1}/b_J < \mu q \le q = s/2 \le \beta(1/2) \le \beta((Tx)_1),$$

where $\beta = \beta_{st}$ is defined in 5.4. Hence $Tx \in F_0$.

Since $TA \subset F_0$, the map $h|A$ is an ε-nearisometry . Since A has the c-IAP, there is an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ with $\|S - h\|_A \le c\varepsilon$. For each $j \in J_k$ we have $(Tu_j)_1 \in \{0, 1\}$. Since $\varphi(0) = \varphi(1) = 0$, this implies that $hu_j = u_j$, and hence $|Su_j - u_j| \le c\varepsilon$, which yields $|P'Su_j| \le c\varepsilon$. Since S is affine, we have

$$Sx = \sum_{j=0}^{k} \xi_j(x) Su_j$$

for all $x \in \mathsf{R}^k$. By 5.14 this implies that $|P'Sx| \le Hc\varepsilon$ for all $x \in Q$ with $H = 2^{k+1} - 1$. Hence

$$|P'SPz| \le Hc\varepsilon.$$

Since $(Tz)_n = |P'z|/b_J$ and $(Tz)_1 = \xi_J(z) = t$, the definition (5.2) of g_φ gives

$$(gTz)_n = (Tz)_n + \varphi((Tz)_1) = |P'z|/b_J + st.$$

Consequently,

$$|P'hz| = b_J|P'gTz| \ge |P'z| + stb_J.$$

On the other hand,

$$|P'hz| \le |P'Sz| + |P'hz - P'Sz| \le |P'Sz| + c\varepsilon.$$

Here

$$|P'Sz| \le |P'SPz| + |P'Sz - P'SPz| \le Hc\varepsilon + |z - Pz| = Hc\varepsilon + |P'z|.$$

Combining the estimates yields

$$st b_J \le (H+1)c\varepsilon = 5 \cdot 2^{k+1} cs^2 t b_J.$$

Since $s = 2q = (2^{2k+5}c)^{-1}$ by (5.11), this implies the contradiction

$$1 \le 5 \cdot 2^{k+1} cs = 5 \cdot 2^{-k-4} \le 5/32.$$

CASE 2: $t_J \le t_J^*$. Now $t_J^* > 0$ and $|P'x| < 4qb_J t_J^*$ for all $x \in A_J$. Moreover, $t_J^* \le H = 2^{k+1} - 1$ by 5.14. Pick $z \in A_J$ with $\xi_J(z) = -t_J^*$. Set $t^* = t_J^*$, $s = 2q$, and let $\varphi^* = \varphi^*_{s,t^*}$: $\mathsf{R} \to \mathsf{R}$ be the function defined in 5.4. Let T: $\mathsf{R}^n \to \mathsf{R}^n$ be the similarity given by 5.17 for these J and z. Let $g^* = g_{\varphi^*}$: $\mathsf{R}^n \to \mathsf{R}^n$ be as in (5.2), and set $h^* = T^{-1} g^* T$. By 5.7, the map $h^*|T^{-1}F_0^*$ is an ε^*-nearisometry with $\varepsilon^* = 6s^2 t^*(t^* \vee 1) b_J$, where F_0^* is defined in 5.4. Since $t^* \le H$, we have $\varepsilon^* \le 6Hs^2 t^* b_J$.

We show that $TA \subset F_0^*$. Let $x \in A$.

SUBCASE 2a: $x \in A_J$. Now

$$(Tx)_1 = \xi_J(x) \le 1/2, \ |(Tx)_n| \le |P'Tx| = |P'x|/b_J < 4qt^* = 2st^*,$$

and hence $Tx \in F_0^*$.

SUBCASE 2b: $x \in A'_J$. Now $(Tx)_1 \ge 1/2$ and

$$|(Tx)_n| \le |P'x|/b_J \le h_{k+1}/b_J \le \alpha h_k/b_J \le q h_k / C_k b_J.$$

By 5.8 and (5.15) this implies that

$$|(Tx)_n| \le q\theta(\Delta)/b_J \le q = s/2,$$

and hence $Tx \in F_0^*$.

Since $A \subset T^{-1}F_0^*$, the map $h^*|A$ is an ε^*-nearisometry . Hence there is an isometry S: $\mathsf{R}^n \to \mathsf{R}^n$ with $\|S - h^*\|_A \le c\varepsilon^*$. Now we can proceed as in Case 1 and obtain $st^* b_J \le 6cH(H+1)s^2 t^* b_J$, which gives the contradiction $1 \le 12c(H+1)^2 q \le 3/4$, which completes the proof of the lemma. ∎

5.19. LEMMA: *If $A \subset \mathsf{R}^k$, then $A = \{u_0, \dots, u_k\}$.*

Proof: Lemma 5.18 implies that $t_J = t_J^* = 0$ for each $J \in \mathcal{J}_k$. Hence $\xi_j(x) \in \{0, 1\}$ for all $0 \le j \le k$. ∎

5.20. LEMMA: *If $x \in A_J$, then*

$$|\xi_J(x)| \le \frac{h_{k+1}}{4qb_J} \le \frac{\mu}{4} \le \frac{1}{4k} \le \frac{1}{4}.$$

Hence $d(A_J, A'_J) \ge b_J/2$, and the sets A_J and A'_J are disjoint.

Proof: Since $|P'y| \le h_{k+1} < \mu q b_J$ for all $y \in A$ by (5.16), the lemma follows from 5.18. ∎

We interpose an elementary result on orthogonal maps. Set $\mathsf{R}^0 = \{0\}$.

5.21. LEMMA: *Suppose that $0 \le p \le n-1$ and that $a \in \mathsf{R}^n \setminus \mathsf{R}^p$. Suppose also that $U\colon \mathsf{R}^n \to \mathsf{R}^n$ is an orthogonal map with $U|\mathsf{R}^p = \mathrm{id}$. Then there is an orthogonal map $T\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $T|\mathsf{R}^p = \mathrm{id}$, $TUa = a$, and*

$$|Tx - x| \le |Ua - a||P'_p x|/|P'_p a|$$

for all $x \in \mathsf{R}^n$, where $P'_p\colon \mathsf{R}^n \to \mathsf{R}^{p\perp}$ is the orthogonal projection.

We next show that for each $J \in \mathcal{J}_k$, one of the sets A_J and A'_J degenerates to a very thin set.

5.22. LEMMA: *For each $J \in \mathcal{J}_K$ we have*

$$A_J \subset \{x \in \mathsf{R}^k \colon \xi_J(x) = 0\} \quad \text{or} \quad A_{J'} \subset \{x \in \mathsf{R}^k : \xi_{J'}(x) = 0\}.$$

Proof: The proof can be regarded as an elaboration of the special case 5.9.2.

Set $\lambda_J = \max\{|P'x| \colon x \in A_J\}$. Then $\lambda_J \ge 4qb_J(t_J \vee t^*_J)$ by 5.18. If $\lambda_J = 0$, this implies that $t_J = t^*_J = 0$, and the lemma follows. The case $\lambda_{J'} = 0$ is similar, and we may thus assume that $\lambda_J > 0$, $\lambda_{J'} > 0$. We show that this leads to a contradiction.

By symmetry, we may assume that $u_{k+1} \in A'_J$. Then

$$\lambda_J \le \lambda_{J'} = h_{k+1} \le \mu q b_J,$$

where the last inequality follows from (5.16).

Pick a point $w \in A_J$ with $|P'w| = \lambda_J$. Define an orthogonal map $U\colon \mathsf{R}^n \to \mathsf{R}^n$ as follows: If $k = n-1$, we set $Ux = (x_1, \ldots, x_{n-1}, -x_n)$. Then

$$U|\mathsf{R}^{n-1} = \mathrm{id}, \quad |Uw - w| = 2\lambda_J. \tag{5.23}$$

If $k \le n-2$, then the sphere $|x| = \lambda_J$ meets $\mathsf{R}^{k\perp} \cap \mathsf{R}^{k+1}$ in two points $\lambda_J e_{k+1}$ and $-\lambda_J e_{k+1}$. Hence there is an orthogonal map $U'\colon \mathsf{R}^{k\perp} \to \mathsf{R}^{k\perp}$ such that

$U'P'w \in \mathsf{R}^{k\perp} \cap \mathsf{R}^{k+1}$ and $|U'P'w - P'w| \geq \lambda_J\sqrt{2}$. Then $U = P + U'P'$ is an orthogonal map of R^n such that

$$U|\mathsf{R}^k = \mathrm{id}, \quad Uw \in \mathsf{R}^{k+1}, \quad |Uw - w| \geq \lambda_J\sqrt{2}. \tag{5.24}$$

Define $f\colon A \to \mathsf{R}^n$ by $f|A_J = U|A_J$ and by $f|A'_J = \mathrm{id}$.

FACT 1: f is an ε-nearisometry with $\varepsilon = 4q\lambda_J$.

To prove this, let $x \in A_J$, $y \in A'_J$, and set $\delta = \big||fx - fy| - |x - y|\big|$. Since $|x - y| \wedge |fx - fy| \geq b_J/2$ by 5.20, we have $\delta b_J \leq \big||fx - fy|^2 - |x - y|^2\big|$. Since $|U'P'x| = |P'x|$, we obtain

$$\begin{aligned}|fx - fy|^2 &= |Ux - y|^2 = |Px - Py|^2 + |U'P'x - P'y|^2 \\ &= |x - y|^2 + 2P'x \cdot P'y - 2U'P'x \cdot P'y,\end{aligned}$$

and hence

$$\delta b_J \leq 4|P'x||P'y| \leq 4\lambda_J h_{k+1} \leq 4\lambda_J q b_J = \varepsilon b_J$$

by (5.16), and Fact 1 follows.

Set

$$\eta = 2c\varepsilon = 8cq\lambda_J.$$

Since A has the c-IAP, there is an isometry $S\colon \mathsf{R}^n \to \mathsf{R}^n$ with $\|S - f\|_A \leq \eta/2$.

FACT 2: There is an orthogonal map $U_{k+1}\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $U_{k+1}|\mathsf{R}^{k+1} = \mathrm{id}$ and $\|U_{k+1} - f\|_A \leq 2^{k+1}\eta$.

We prove Fact 2 by induction by constructing for each integer $i \in [1, k+1]$ an orthogonal map U_i of R^n such that

$$U_i|\mathsf{R}^i = \mathrm{id}, \quad \|U_i - f\|_A \leq 2^i\eta. \tag{5.25}$$

Since $f|\mathsf{R}^k \cap A = \mathrm{id}$ and since $u_{k+1} \in A'_J$, we have

$$f|\{u_0, \ldots, u_{k+1}\} = \mathrm{id}.$$

Setting $Tx = Sx - S(0)$ we get an orthogonal map of R^n. Since $S(0) \leq \eta/2$, we have $\|T - f\|_A \leq \eta$. Since $|Tu_1 - u_1| = |Tu_1 - fu_1| \leq \eta$, there is an orthogonal map T_1 of R^n such that $T_1Tu_1 = u_1$ and such that $|T_1x - x| \leq \eta|x|$ for all $x \in \mathsf{R}^n$. Setting $U_1 = T_1T$ we thus have $\|U_1 - T\|_A \leq \eta$, which implies $\|U_1 - f\|_A \leq 2\eta$. The case $i = 1$ of (5.25) is proved.

Assume that $1 \leq p \leq k$ and that we have found maps $U_1, \ldots, U_p$ satisfying (5.25). Then $|U_pu_{p+1} - u_{p+1}| \leq 2^p\eta$. By 5.21 there is an orthogonal map T_{p+1} of R^n such that

$$T_{p+1}|\mathsf{R}^p = \mathrm{id}, \quad T_{p+1}U_pu_{p+1} = u_{p+1}, \quad \text{and} \quad |T_{p+1}x - x| \leq 2^p\eta|P'_px|/h_{p+1}$$

for all $x \in \mathrm{R}^n$, where P'_p is the orthogonal projection onto $\mathrm{R}^{p\perp}$. Setting $U_{p+1} = T_{p+1}U_p$ we have $U_{p+1}|\mathrm{R}^{p+1} = \mathrm{id}$. For $x \in A$ we have

$$|U_{p+1}x - fx| \le |T_{p+1}U_p x - U_p x| + |U_p x - fx| \le 2^p\eta|P'_p U_p x|/h_{p+1} + 2^p\eta.$$

Since $|P'_p U_p x| = |P'_p x| \le h_{p+1}$, we obtain (5.25) for $i = p+1$, and Fact 2 is proved.

To complete the proof of the lemma, we first assume that $k \le n-2$. Since $fw = Uw \in \mathrm{R}^{k+1}$, we have $fw = U_{k+1}Uw$, and hence

$$|U_{k+1}w - fw| = |U_{k+1}w - U_{k+1}Uw| = |w - Uw| \ge \lambda_J\sqrt{2}$$

by (5.24). Since $\|U_{k+1} - f\|_A \le 2^{k+1}\eta = 2^{k+4}cq\lambda_J$ by Fact 2, this yields the contradiction

$$\sqrt{2} \le 2^{k+4}cq = 2^{-k-2} \le 1/8.$$

Finally, let $k = n-1$. Now $U_n = U_n|\mathrm{R}^n = \mathrm{id}$ by Fact 2. Since $|w - fw| = 2\lambda_J$ by (5.23), Fact 2 implies that $2\lambda_J \le 2^n\eta = 2^{n+3}cq\lambda_J$, which gives the contradiction $2 \le 2^{-n-1}$. ∎

For $i \in J_k$ we write $A_i = A_{\{i\}}$ and $A'_i = A'_{\{i\}}$.

5.26. Lemma: *Let $J, K \in \mathcal{J}_k$.*

(1) *If $J \subset K$, then $A_J \supset A_K$ and $A'_J \subset A'_K$.*
(2) *If $J \cap K = \varnothing$, then $A'_J \cap A'_K = \varnothing$.*
(3) $A = \bigcup_{j=0}^k A'_j$.
(4) $A'_i = \bigcap_{j\ne i} A_j$ *for each $i \in J_k$.*
(5) $A'_i \ne \{u_i\}$ *for at most one $i \in J_k$.*

Proof: (1) If $x \in A_K$, then $\xi_J(x) = \xi_K(x) - \xi_{K\setminus J}(x)$. By 5.20 this implies that $\xi_J(x) \le 1/4k + 1/4k \le 1/2$, and hence $x \in A_J$. This proves (1), since $A'_J = A \setminus A_J$.

(2) Since $J \subset K'$, (1) implies that $A'_J \subset A'_{K'} = A \setminus A'_K$.

(3) If $x \in A$, there is $j \in J_k$ with $\xi_j(x) \ge 1/(k+1) > 1/4k$. By 5.20 this implies that $x \in A'_j$.

(4) If $j \ne i$, then $A'_i \cap A'_j = \varnothing$ by (2), and hence $A'_i \subset A_j$. Conversely, (3) implies that $\bigcap_{j\ne i} A_j = \bigcup_{j=0}^k A'_j \setminus \bigcup_{j\ne i} A'_j \subset A'_i$.

(5) If $A \subset \mathrm{R}^k$, then $A = \{u_0, \ldots, u_k\}$ by 5.19. Assume that $A \not\subset \mathrm{R}^k$. By (3) there is $i \in J_k$ with $A'_i \not\subset \mathrm{R}^k$. By 5.22 we have $A_i \subset \mathrm{R}^k$, and $\xi_i(y) = 0$ for all $y \in A_i$.

Let $j \neq i$ and let $x \in A'_j$. It suffices to show that $x = u_j$. Now $x \in A_i$ by (4), and hence $\xi_i(x) = 0$. If $k = 1$, this implies that $x = u_j$. Assume that $k \geq 2$ and choose $\nu \in J_k$ with $i \neq \nu \neq j$. It suffices to show that $\xi_\nu(x) = 0$.

Since $j \in \{i, \nu\}'$, we have $x \in A_{\{i,\nu\}}$ by (1). Moreover, $A'_i \subset A'_{\{i,\nu\}}$, and hence $A'_{\{i,\nu\}} \not\subset \mathsf{R}^k$. By 5.22 this implies that $x \in \mathsf{R}^k$ and that $0 = \xi_i(x) + \xi_\nu(x) = \xi_\nu(x)$. ∎

5.27. Lemma: *For each $i \in J_k$ we have $A'_i \subset \bar{B}(u_i, Mh_{k+1})$, where $M = 1 + kc'_1 \cdots c'_k/4q$ is as in* (5.11).

Proof: Let $x \in A'_i$. By 5.20 we have $|1 - \xi_i(x)| \leq h_{k+1}/4qb_i$. Moreover, if $j \neq i$, then $x \in A_j$ by 5.26(4), and 5.20 yields $|\xi_j(x)| \leq h_{k+1}/4qb_j$. Thus

$$|x - u_i| \leq \sum_{j \neq i} |\xi_j(x)||u_j| + |1 - \xi_i(x)||u_i| + |P'x| \leq h_{k+1}\Big(1 + \frac{1}{4q}\sum_{j=1}^{k} \frac{|u_j|}{b_j}\Big).$$

Since (P_ν) holds for $1 \leq \nu \leq k$, we have $|u_\nu| \leq c'_\nu h_\nu$ for these ν. Now Lemma 2.9 gives $|u_j|/b_j \leq c'_1 \cdots c'_k$, and the lemma follows. ∎

5.28. *Proof of* 5.1 *continues.*

Case 1: A is infinite. Now $A \not\subset \mathsf{R}^k$ by 5.19. By 5.26(5), there is a unique $i \in J_k$ with $A'_i \neq \{u_i\}$. For each $j \in J_k$, the set $A'_j = A \setminus A_j$ is a neighborhood of u_j in A. Hence the points u_j, $j \neq i$, are isolated in A. Since u_0 is a cluster point, we have $i = 0$. Moreover, 5.27 gives $A \setminus \{u_1, \dots, u_k\} \subset \bar{B}(Mh_{k+1})$. Hence (P_{k+1}) holds with $c'_{k+1} = M$.

Case 2: A is finite. If $A \subset \mathsf{R}^k$, then (P_{k+1}) follows from 5.19 with $c'_{k+1} = 1$. Assume that $A \not\subset \mathsf{R}^k$. As in Case 1, we find $i \in J_k$ with $A'_i \neq \{u_i\}$. It suffices to show that $i = 0$, since (P_{k+1}) will then follow from 5.27 with $c'_{k+1} = M$.

Assume that $i \neq 0$. By 5.27 we have $A'_i \subset \bar{B}(u_i, r)$ with $r = Mh_{k+1}$. Choose a point $x \in A'_i$ with $x \neq u_i$. Since $d(u_0, A \setminus \{u_0\})$ is minimal, there is $y \in A$ such that $y \neq u_0$ and $|y - u_0| \leq |x - u_i| \leq r$. If $y \in A'_i$, then $|u_0 - u_i| \leq |u_0 - y| + |y - u_i| \leq 2r$. If $y \notin A'_i$, then $y = u_j$ for some $j \notin \{0, i\}$, and hence $|u_0 - u_j| \leq r$. In both cases we have found $j \neq 0$ with $|u_j| = |u_0 - u_j| \leq 2r$. Hence $h_k \leq h_j \leq |u_j| \leq 2r = 2Mh_{k+1}$.

On the other hand, $h_{k+1} < \alpha h_k = \mu q h_k / C_k$ by (5.13), and $\mu \leq C_k/3Mq$ by (5.12). These inequalities yield the contradiction $h_k < 2h_k/3$, and Theorems 5.1 and 2.5 are proved. ∎ ∎

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